J. Phys. A: Math. Gen. **32** (1999) 5329–5342. Printed in the UK PII: S0305-4470(99)98660-2

Relativistic two-particle one-dimensional scattering problem for superposition of δ-potentials

V N Kapshai and T A Alferova

Department of Physics, Gomel State University, Sovetskaia Str., 102, Gomel, 246699, Belarus

Received 26 October 1998, in final form 23 April 1999

Abstract. The covariant single-time equations of the quantum field theory are formulated in the relativistic configurational representation. The explicit formulae for the Green functions corresponding to the scattering states are calculated in this representation. Using the derived nonhomogeneous equations the scattering problem is solved exactly for certain potentials (combinations of zero-range potentials). The equations and their solutions are studied in the non-relativistic limit. The conditions of total reflection, available for such potentials, are investigated.

1. Introduction

In the momentum representation the quasipotential equations for the wavefunction of the two-particle system [1, 2] (in this paper we consider one space dimension) are analogous to the Schrödinger equation

$$\Psi_q(p) = 2\pi\delta(p-q) + G_q^{(0)}(p)\int V(p,k)\Psi_q(k)\frac{\mathrm{d}k}{2\pi} \tag{1}$$

where $(V = mU)$

$$G_q^{(0)}(p) = \frac{1}{q^2 - p^2 + \mathrm{i}0} \tag{2}$$

and the relativistic equations for the two-particle scattering amplitude are analogous to the non-relativistic Lippman–Schwinger equation. It is well known that in the non-relativistic theory the potential $V(p,k)$ is usually a local one $(V(p,k) = V(p-k))$, so the Schrödinger equation can easily be derived in the coordinate representation, where it is usually written in the differential form.

The direct and inverse Fourier transforms

$$\begin{aligned}\Psi_q(p) &= \int \exp(-\mathrm{i}px)\Psi_q(x)\,\mathrm{d}x\\ V(p-k) &= \int \exp[-\mathrm{i}(p-k)x]V(x)\,\mathrm{d}x\end{aligned} \tag{3}$$

establish the unitary equivalence between the coordinate and momentum representations. However, for the relativistic equations Fourier analysis is of no use, since the equations in the coordinate representation are integro-differential (non-local) [3, 4]. First of all, relativistic potentials (quasipotentials) $V(p,k)$ are not local; second, relativistic Green functions in the momentum representation $G_m^{(j)}(E_q; p)$ contain the square root $E_p = \sqrt{p^2+m^2}$, where m is the mass of the particle $(m_1 = m_2 = m)$. For example, the Logunov–Tavkhelidze equation for

0305-4470/99/285329+14$30.00

two scalar particles ($j = 1$) and the Kadyshevsky equation for two spinor particles ($j = 2$) contain the following Green functions:

$$G_m^{(1)}(E_q; p) = \frac{1}{E_q^2 - E_p^2 + \mathrm{i}0} \cdot \frac{m}{E_p} \qquad G_m^{(2)}(E_q; p) = \frac{1}{2E_q - 2E_p + \mathrm{i}0} \cdot \frac{m}{E_p^2} \tag{4}$$

respectively. In the latter expression $2E_q$ is the energy of two particles in the centre-of-mass system, which for scattering states ($2E_q > 2m$) can be parametrized as follows: $2E_q = 2\sqrt{q^2 + m^2}$. We also consider the modified Logunov–Tavkhelidze ($j = 3$)† and modified Kadyshevsky ($j = 4$) equations, where

$$G_m^{(3)}(E_q; p) = \frac{1}{E_q^2 - E_p^2 + \mathrm{i}0} \qquad G_m^{(4)}(E_q; p) = \frac{1}{2E_q - 2E_p + \mathrm{i}0} \cdot \frac{1}{E_p}. \tag{5}$$

It is easy to see that all four relativistic Green functions, (4) and (5), in the non-relativistic limit ($m \to \infty$) transform into (2)

$$\lim_{m\to\infty} G_m^{(j)}(E_q; p) = G_q^{(0)}(p) \qquad j = 1\text{–}4. \tag{6}$$

Hence, the quasipotential equations ($j = 1$–4)

$$\Psi_q^{(j)}(p) = 2\pi\delta(p - q) + G_m^{(j)}(E_q; p) \int V(p, k)\Psi_q^{(j)}(k)\frac{\mathrm{d}k}{2\pi} \tag{7}$$

in the non-relativistic limit are transformed into (1). In addition, the quasipotential wavefunction $\Psi_q^{(j)}(p)$ in the one-dimensional space has only one component both for the system of scalar particles and for the system of spinor particles, since in the latter case the wavefunction should be projected on the states with positive energy.

2. Equations in the relativistic configurational representation

Although relativistic quasipotentials are not local they possess the following important property. For example, the quasipotential of one-boson exchange for the completely scalar equations ($j = 1, 3$) has the following form:

$$V(p, k) \cong \frac{1}{\mu^2 - (p_{(2)} - k_{(2)})^2 - \mathrm{i}0} = \frac{1}{\mu^2 - 2m^2 + 2p_{(2)}k_{(2)} - \mathrm{i}0} \tag{8}$$

where $p_{(2)}$ and $k_{(2)}$, the initial and the final two-momenta of the first particle, are given by

$$p_{(2)} = (p_{(2)}^0, p_{(2)}^1) = (E_p, p) = (m \cosh \chi_p, m \sinh \chi_p) \tag{9}$$

where χ_p is rapidity, since $p_{(2)}$ and $k_{(2)}$ are on mass 'hyperbola'‡. Consequently, $p_{(2)}k_{(2)} = m^2 \cosh(\chi_p - \chi_k)$ and potential (8) depends on the difference between the rapidities $V(p, k) = V(\chi_p - \chi_k)$. The three-dimensional quasipotentials possess the same property [5, 6].

This makes it possible to carry out the transformation to the relativistic configuration representation (RCR) instead of the transformation to the coordinate representation. The RCR is introduced with the help of expansion in the principal series of the unitary representation of the Lorentz group [5, 6]. In the three-dimensional space it is equivalent to the expansion in the functions:

$$\xi(\vec{r}, \vec{p}) = \left(\frac{E_p - \vec{p}\vec{n}}{m}\right)^{-1-\mathrm{i}mr} \qquad m > 0 \qquad 0 \leqslant r < \infty. \tag{10}$$

† It should be noted that the relativistic Green function $G^{(3)}$ is similar to the non-relativistic $G^{(0)}$.
‡ We only consider the case when $m > 0$.

In the one-dimensional space the transformation to the RCR is realized as the expansion in the following functions:

$$\xi(\rho, p) = \left(\frac{E_p - p}{m}\right)^{-\mathrm{i}m\rho} = \exp(\mathrm{i}\chi_p m\rho) \qquad m > 0 \qquad -\infty < \rho < \infty. \tag{11}$$

Here ρ is the relativistic relative coordinate which is canonically conjugate to the rapidity χ_p multiplied by mass m. Since

$$\lim_{m\to\infty} m\chi_p = \lim_{m\to\infty} m \operatorname{arcsinh}\left(\frac{p}{m}\right) = p \qquad \lim_{m\to\infty} \xi(\rho, p) = \exp(\mathrm{i}p\rho) \tag{12}$$

then in the non-relativistic limit ρ transforms into the ordinary coordinate x which is canonically conjugate to the momentum p. We should emphasize here that the expansion in functions (11) is only possible in the case when $m > 0$, that is, when the two-momenta of particles are on the mass 'hyperbola'. When the mass is null the transformation to the RCR is impossible and therefore, in this paper, we do not consider the ultra-relativistic case†.

Equations in the RCR, as a rule, are finite-difference ones with the local potential [5, 6]. However, the investigation of the finite-difference equations, especially for the singular potentials, is a difficult problem. In addition, as is well known in the general case, every solution of these equations contains arbitrary 'i-periodic multipliers'. Formulation of integral equations in the RCR [7] gives us the possibility:

(i) to preserve the physically obvious description of potentials;
(ii) to get rid of the i-periodic multipliers;
(iii) to consider singular δ-potentials.

There is the same possibility if the non-relativistic integral equations are written in the coordinate representation. Thus, (1) with the help of (3) can be transformed into

$$\Psi_q(x) = \exp(\mathrm{i}qx) + \int G_q^{(0)}(x, y)V(y)\Psi_q(y)\,\mathrm{d}y \tag{13}$$

where

$$G_q^{(0)}(x, y) = \frac{1}{2\pi}\int G_q^{(0)}(p)\exp[\mathrm{i}p(x-y)]\,\mathrm{d}p = \frac{-\mathrm{i}}{2q}\exp(\mathrm{i}q|x-y|) \tag{14}$$

is the non-relativistic Green function for the continuous spectrum.

Let us carry out the following transformation:

$$\Psi_q^{(j)}(\rho) = \int \exp(\mathrm{i}\chi_p m\rho)\Psi_q^{(j)}(p)\frac{\mathrm{d}p}{2\pi} = \int \exp(\mathrm{i}\chi_p m\rho)\tilde{\Psi}_q^{(j)}(\chi_p)\frac{m\,\mathrm{d}\chi_p}{2\pi} \tag{15}$$

$$\tilde{\Psi}_q^{(j)}(\chi_p) = \Psi_q^{(j)}(p)\frac{E_p}{m} = \int \exp(-\mathrm{i}\chi_p m\rho')\Psi_q^{(j)}(\rho')\,\mathrm{d}\rho' \tag{16}$$

in (7) when $V(p, k)$ is the following scalar quasipotential

$$V(\chi_p - \chi_k) = \int \exp[-\mathrm{i}(\chi_p - \chi_k)m\rho']V(\rho')\,\mathrm{d}\rho'. \tag{17}$$

Then the equation for the wavefunction $\Psi_q^{(j)}(\rho)$ in the RCR can be written as follows:

$$\Psi_q^{(j)}(\rho) = \exp(\mathrm{i}\chi_p m\rho) + \int G_m^{(j)}(E_q; \rho, \rho')V(\rho')\Psi_q^{(j)}(\rho')\,\mathrm{d}\rho'. \tag{18}$$

The relativistic Green functions in the RCR are defined as follows:

$$G_m^{(j)}(E_q; \rho, \rho') = \int \exp[\mathrm{i}\chi_p m(\rho - \rho')]G_m^{(j)}(E_q; p)\frac{\mathrm{d}p}{2\pi}. \tag{19}$$

† Equations (7) with the Green functions (4) and (5) are not well defined when the mass is null as the Green functions contain the following factors: $|p|^{-1}$; $(E_q - |p|)^{-1}$.

It is not difficult to see that the following relation:

$$\lim_{m\to\infty} G_m^{(j)}(E_q;\rho,\rho') = \int \exp[\mathrm{i}p(\rho-\rho')]G_q^{(0)}(p)\frac{\mathrm{d}p}{2\pi} = G_q^{(0)}(\rho,\rho') \qquad (20)$$

holds for all four functions (4) and (5) in the non-relativistic limit. In order to calculate the functions $G_m^{(j)}(E_q;\rho,\rho')$ it is necessary to integrate (19) over χ_q. Then for $j=1$ we have the expression

$$G_m^{(1)}(E_q;\rho,\rho') = \frac{1}{2\pi m}\int \frac{\exp[\mathrm{i}\chi_p m(\rho-\rho')]}{\cosh^2\chi_q - \cosh^2\chi_p + \mathrm{i}0}\mathrm{d}\chi_p \qquad (21)$$

and analogous ones for all other cases. Explicitly, formulae for the Green functions $G_m^{(j)}(E_q;\rho,\rho')$ can be obtained using a technique of contour integration

$$G_m^{(1)}(E_q;\rho,\rho') = \frac{-\mathrm{i}}{K_q^{(1)}}\frac{\sinh[(\frac{\pi}{2}+\mathrm{i}\chi_q)m(\rho-\rho')]}{\sinh[\frac{\pi}{2}m(\rho-\rho')]} \qquad (22)$$

$$G_m^{(2)}(E_q;\rho,\rho') = \frac{-\mathrm{i}}{K_q^{(2)}}\frac{\sinh[(\pi+\mathrm{i}\chi_q)m(\rho-\rho')]}{\sinh[\pi m(\rho-\rho')]} + \frac{(4m\cosh\chi_q)^{-1}}{\cosh[\frac{\pi}{2}m(\rho-\rho')]} \qquad (23)$$

$$G_m^{(3)}(E_q;\rho,\rho') = \frac{-\mathrm{i}}{K_q^{(3)}}\frac{\cosh[(\frac{\pi}{2}+\mathrm{i}\chi_q)m(\rho-\rho')]}{\cosh[\frac{\pi}{2}m(\rho-\rho')]} \qquad (24)$$

$$G_m^{(4)}(E_q;\rho,\rho') = \frac{-\mathrm{i}}{K_q^{(4)}}\frac{\sinh[(\pi+\mathrm{i}\chi_q)m(\rho-\rho')]}{\sinh[\pi m(\rho-\rho')]}. \qquad (25)$$

In formulae (22)–(25) and below, the following notations have been used:

$$K_q^{(1)} = K_q^{(2)} = m\sinh 2\chi_q \qquad K_q^{(3)} = K_q^{(4)} = 2m\sinh\chi_q. \qquad (26)$$

Since $\lim_{m\to\infty} K_q^{(j)} = K_q^{(0)} = 2q$, we see, using (22)–(25) that the limit relation (20) holds and hence (18) transforms into (13) in the non-relativistic limit. It should be noted that if the relativistic quasipotential depends on the mass ($V = V_m(\rho)$) then the non-relativistic potential is defined by $\lim_{m\to\infty} V_m(\rho)$. For example, the quasipotential [7]

$$V_m(\rho) = \frac{g}{\rho}\tanh\left(\frac{\pi m}{2}\rho\right) \qquad (27)$$

(some superposition of one-boson exchange potentials) in the non-relativistic limit transforms into the Coulomb potential.

3. Solutions of the relativistic equations with δ-potentials

Our programme for the future includes the investigation of the considered equations with the quasipotentials derived on the basis of quantum field theory, for instance, with (27). Therefore, it would be attractive to construct the models of two-particle problems which are solved exactly. It is clear that such models should have the non-relativistic limit. In this case, above all, we should consider the potentials for which both the non-relativistic problem and, if it is possible, the one-particle relativistic problem, i.e. the Dirac equation, are solved exactly.

In the past models of point (or contact) interaction have been much developed in non-relativistic quantum mechanics (see monographs [8–10]). At the same time the one-dimensional Schrödinger equation with point interactions or their generalizations (such as the Kronig–Penney model with a periodical superposition of δ-potentials) [11–13] have also attracted a lot of attention.

The three-dimensional and one-dimensional Dirac equations with δ-potentials have been studied recently [14–19]. The one-dimensional Dirac equation has been discussed aiming to study models with superposition of δ-potentials: the Kronig–Penney model [20] and the relativistic Tamm model [21].

Our aim is to investigate relativistic two-particle equations (18) and (22)–(25) with δ-potentials, since exact solutions of all four equations can be obtained. The following important question arises in this context: 'what kind of quasipotential equation is preferable?'. However, this problem has so many 'degrees of freedom' that we cannot answer this question precisely [22].

So, to start let us consider equations (18) with

$$V(\rho) = V\delta(\rho) \tag{28}$$

(V—is real). The solution of (18) with (28) is given by

$$\Psi_q^{(j)}(\rho) = \exp(\mathrm{i}\chi_q m\rho) + G_m^{(j)}(E_q;\rho,0)V\Psi_q^{(j)}(0) \tag{29}$$

$$\Psi_q^{(j)}(0) = [1 - G_m^{(j)}(E_q)V]^{-1}. \tag{30}$$

In (30) we use the following notations:

$$G_m^{(j)}(E_q) = G_m^{(j)}(E_q;\rho,\rho) = \frac{-\mathrm{i}}{K_q^{(j)}}[1 + \mathrm{i}\beta_q^j] \tag{31}$$

where for all the Green functions considered, (22)–(25), β_q^j are as follows:

$$\beta_q^1 = \frac{2\chi_q}{\pi} \qquad \beta_q^2 = \frac{\chi_q}{\pi} + \frac{\sinh\chi_q}{2} \qquad \beta_q^3 = 0 \qquad \beta_q^4 = \frac{\chi_q}{\pi} \tag{32}$$

and in the non-relativistic limit (q is fixed, $m \to \infty$) $\lim_{m\to\infty}\beta_q^j = 0$ so that

$$\lim_{m\to\infty} G_m^{(j)}(E_q) = \frac{-\mathrm{i}}{2q} = \frac{-\mathrm{i}}{K_q^{(0)}}. \tag{33}$$

In order to obtain physical information about the penetration and reflection probabilities let us consider the asymptotic behaviour of the wavefunctions (29) for $\rho\pm \to \infty$. It is not difficult to see that all the Green functions, (22)–(25), have the following limit behaviour:

$$G_m^{(j)}(E_q;\rho,\rho')|_{\rho\to\pm\infty} = \frac{-\mathrm{i}}{K_q^{(j)}}\exp[\pm\mathrm{i}\chi_q m(\rho-\rho')]. \tag{34}$$

The asymptotic formulae for wavefunctions are

$$\Psi_q^{(j)}(\rho)|_{\rho\to\infty} = \exp(\mathrm{i}\chi_q m\rho) + A_q^{(j)}\exp(\mathrm{i}\chi_q m\rho) \tag{35}$$

$$\Psi_q^{(j)}(\rho)|_{\rho\to-\infty} = \exp(\mathrm{i}\chi_q m\rho) + B_q^{(j)}\exp(-\mathrm{i}\chi_q m\rho). \tag{36}$$

The amplitude coefficients A and B (for single δ-potential (28) they are the same) are given by

$$A_q^{(j)} = B_q^{(j)} = \frac{-\mathrm{i}}{K_q^{(j)}}\cdot\frac{V}{1 - G_m^{(j)}(E_q)V}. \tag{37}$$

Coefficients A and B are in agreement with the non-relativistic limit (q is fixed, $m \to \infty$), where

$$A_q^{(0)} = B_q^{(0)} = \frac{-\mathrm{i}V}{2q + \mathrm{i}V}. \tag{38}$$

Another important property of the coefficients A and B is that the penetration and reflection coefficients, defined by

$$P_q^{(j)} = |1 + A_q^{(j)}|^2 \qquad R_q^{(j)} = |B_q^{(j)}|^2 \tag{39}$$

in all four cases comply with the relativistic unitary relation

$$|1 + A_q^{(j)}|^2 + |B_q^{(j)}|^2 = 1. \tag{40}$$

Let us now consider the following superposition of δ-potentials ($V_{1,2}$—are real)

$$V(\rho) = V_1\delta(\rho - a) + V_2\delta(\rho + a). \tag{41}$$

The wavefunctions of equations (18) with the potential (41) are

$$\Psi_q^{(j)}(\rho) = \exp(\mathrm{i}\chi_q m\rho) + G_m^{(j)}(E_q;\rho,a)V_1\Psi_q^{(j)}(a) + G_m^{(j)}(E_q;\rho,-a)V_2\Psi_q^{(j)}(-a). \tag{42}$$

The asymptotic behaviour of the wavefunctions (42) is given by (35) and (36) as well, but in this case the amplitude coefficients are defined by

$$A_q^{(j)} = \frac{-\mathrm{i}}{K_q^{(j)}}[\exp(-\mathrm{i}\chi_q ma)V_1\Psi_q^{(j)}(a) + \exp(\mathrm{i}\chi_q ma)V_2\Psi_q^{(j)}(-a)], \tag{43}$$

$$B_q^{(j)} = \frac{-\mathrm{i}}{K_q^{(j)}}[\exp(\mathrm{i}\chi_q ma)V_1\Psi_q^{(j)}(a) + \exp(-\mathrm{i}\chi_q ma)V_2\Psi_q^{(j)}(-a)]. \tag{44}$$

The constants $\Psi_q^{(j)}(a)$ and $\Psi_q^{(j)}(-a)$ should be solutions of the following algebraic system:

$$\begin{bmatrix} 1 - G_m^{(j)}(E_q)V_1 & -G_m^{(j)}(E_q;a,-a)V_2 \\ -G_m^{(j)}(E_q;-a,a)V_1 & 1 - G_m^{(j)}(E_q)V_2 \end{bmatrix} \begin{bmatrix} \Psi_q^{(j)}(a) \\ \Psi_q^{(j)}(-a) \end{bmatrix} = \begin{bmatrix} \exp(\mathrm{i}\chi_q ma) \\ \exp(-\mathrm{i}\chi_q ma) \end{bmatrix}. \tag{45}$$

Solutions of this system are defined as follows:

$$\Psi_q^{(j)}(a) = \frac{\Delta_{1q}^{(j)}(a)}{\Delta_q^{(j)}(a)} \qquad \Psi_q^{(j)}(-a) = \frac{\Delta_{2q}^{(j)}(a)}{\Delta_q^{(j)}(a)} \tag{46}$$

where

$$\Delta_{1q}^{(j)}(a) = \exp(\mathrm{i}\chi_q ma)[1 - G_m^{(j)}(E_q)V_2] + \exp(-\mathrm{i}\chi_q ma)G_m^{(j)}(E_q;a,-a)V_2 \tag{47}$$

$$\Delta_{2q}^{(j)}(a) = \exp(-\mathrm{i}\chi_q ma)[1 - G_m^{(j)}(E_q)V_1] + \exp(\mathrm{i}\chi_q ma)G_m^{(j)}(E_q;-a,a)V_1 \tag{48}$$

and $\Delta_q^{(j)}(a)$ is the principle determinant of (45). It is given by the following general expression:

$$\Delta_q^{(j)}(a) = \prod_{s=1}^{2}[1 - G_m^{(j)}(E_q)V_s] - [G_m^{(j)}(E_q;a,-a)]^2 V_1V_2 \tag{49}$$

and

$$\tilde{\Delta}_q^{(j)}(a) = (K_q^{(j)})^2\Delta_q^{(j)}(a). \tag{50}$$

For each of the considered Green functions $\tilde{\Delta}_q^{(j)}(a)$ can be written as follows:

$$\begin{aligned}
\tilde{\Delta}_q^{(1)}(a) &= \prod_{s=1}^{2}\left[K_q^{(1)} + \left(\mathrm{i} - 2\frac{\chi_q}{\pi}\right)V_s\right] + \frac{\sinh^2(\alpha + \mathrm{i}\tilde{\chi}_q)}{\sinh^2\alpha}V_1V_2 \\
\tilde{\Delta}_q^{(2)}(a) &= \prod_{s=1}^{2}\left[K_q^{(2)} + \left(\mathrm{i} - \frac{\chi_q}{\pi} - \frac{\sinh\chi_q}{2}\right)V_s\right] - \left[\frac{\mathrm{i}\sinh(2\alpha + \mathrm{i}\tilde{\chi}_q)}{\sinh 2\alpha} - \frac{\sinh\chi_q}{2\cosh\alpha}\right]^2 V_1V_2 \\
\tilde{\Delta}_q^{(3)}(a) &= \prod_{s=1}^{2}[K_q^{(3)} + \mathrm{i}V_s] + \frac{\cosh^2(\alpha + \mathrm{i}\tilde{\chi}_q)}{\cosh^2\alpha}V_1V_2 \\
\tilde{\Delta}_q^{(4)}(a) &= \prod_{s=1}^{2}\left[K_q^{(4)} + \left(\mathrm{i} - \frac{\chi_q}{\pi}\right)V_s\right] + \frac{\sinh^2(2\alpha + \mathrm{i}\tilde{\chi}_q)}{\sinh^2 2\alpha}V_1V_2.
\end{aligned} \tag{51}$$

Here and later, for brevity, we use the following notations:

$$\tilde{\chi}_q = 2\chi_q ma \qquad \alpha = \pi ma. \tag{52}$$

In the non-relativistic approximation (q; a are fixed, $m \to \infty$) all expressions (51) tend to the same limit:

$$\Delta_q^{(0)}(a) = \lim_{m\to\infty} \Delta_q^{(j)}(a) = \frac{1}{4q^2}[(2q + \mathrm{i}V_1)(2q + \mathrm{i}V_2) + \exp(\mathrm{i}4qa)V_1V_2]. \quad (53)$$

The amplitude coefficient of transient and reflected, A and B, waves, are given by

$$A_q^{(j)} = \frac{\tilde{\Delta}_{Aq}^{(j)}(a)}{\tilde{\Delta}_q^{(j)}(a)} \qquad B_q^{(j)} = \frac{\tilde{\Delta}_{Bq}^{(j)}(a)}{\tilde{\Delta}_q^{(j)}(a)}. \quad (54)$$

For the transient wave $\tilde{\Delta}_{Aq}^{(j)}(a)$ is given by

$$\mathrm{i}\tilde{\Delta}_{Aq}^{(j)}(a) = K_q^{(j)}(V_1 + V_2) + D_{Aq}^{(j)}(a)V_1V_2 \quad (55)$$

$$D_{Aq}^{(j)}(a) = 2K_q^{(j)}[G_m^{(j)}(E_q; a, -a)\cos\tilde{\chi}_q - G_m^{(j)}(E_q)]. \quad (56)$$

Since we intend to carry out numerical analysis of the penetration and reflection coefficients, let us present here, explicit expressions for (56)

$$\begin{aligned} D_{Aq}^{(1)}(a) &= 2\mathrm{i}\sin^2\tilde{\chi}_q + \coth\alpha\sin 2\tilde{\chi}_q - 4\pi^{-1}\chi_q \\ D_{Aq}^{(2)}(a) &= D_{Aq}^{(4)}(a) + \sinh\chi_q((\cosh\alpha)^{-1}\cosh\tilde{\chi}_q - 1) \\ D_{Aq}^{(3)}(a) &= 2\mathrm{i}\sin^2\tilde{\chi}_q + \tanh\alpha\sin 2\tilde{\chi}_q \\ D_{Aq}^{(4)}(a) &= 2\mathrm{i}\sin^2\tilde{\chi}_q + \coth 2\alpha\sin 2\chi_q - 2\pi^{-1}\chi_q \end{aligned} \quad (57)$$

where $\tilde{\chi}_q$ and α are given by (52). In the non-relativistic limit $D_{Aq}^{(j)}(a)$ ($j = 1$–4) transforms into

$$D_{Aq}^{(0)}(a) = \lim_{m\to\infty} D_{Aq}^{(j)}(a) = 2\mathrm{i}\sin^2 2qa + \sin 4qa. \quad (58)$$

For the reflected wave $\tilde{\Delta}_{Bq}^{(j)}(a)$ is given by

$$\mathrm{i}\tilde{\Delta}_{Bq}^{(j)}(a) = K_q^{(j)}[V_1\exp(\mathrm{i}\tilde{\chi}_q) + V_2\exp(-\mathrm{i}\tilde{\chi}_q)] + D_{Bq}^{(j)}(a)V_1V_2 \quad (59)$$

$$D_{Bq}^{(j)}(a) = 2K_q^{(j)}[G_m^{(j)}(E_q; a, -a) - G_m^{(j)}(E_q)\cos\tilde{\chi}_q]. \quad (60)$$

The explicit expressions for $D_{Bq}^{(j)}(a)$ ($j = 1$–4) are given by

$$\begin{aligned} D_{Bq}^{(1)}(a) &= 2\coth\alpha\sin\tilde{\chi}_q - 4\pi^{-1}\chi_q\cos\tilde{\chi}_q \\ D_{Bq}^{(2)}(a) &= D_{Bq}^{(4)}(a) - \sinh\chi_q\cos\tilde{\chi}_q + \sinh\chi_q(\cosh\alpha)^{-1} \\ D_{Bq}^{(3)}(a) &= 2\tanh\alpha\sin\tilde{\chi}_q \\ D_{Bq}^{(4)}(a) &= 2\coth 2\alpha\sin\tilde{\chi}_q - 2\pi^{-1}\chi_q\cos\tilde{\chi}_q. \end{aligned} \quad (61)$$

The following non-relativistic limit holds

$$D_{Bq}^{(0)}(a) = \lim_{m\to\infty} D_{Bq}^{(j)}(a) = 2\sin(2qa). \quad (62)$$

Thus, both general expressions for scattering characteristics and their explicit forms have been calculated for all four quasipotential equations. These expressions tend to the same limit in the non-relativistic approximation and the amplitude coefficients of transient and reflected waves are given by†

$$1 + A_q^{(0)} = \frac{4q^2}{4q^2 - 2V_1V_2\sin^2 2qa + \mathrm{i}[2q(V_1 + V_2) + V_1V_2\sin 4qa]} \quad (63)$$

$$B_q^{(0)} = -2\mathrm{i}\cdot\frac{q(V_1 + V_2)\cos 2qa + \mathrm{i}q(V_1 - V_2)\sin 2qa + V_1V_2\sin 2qa}{4q^2 - 2V_1V_2\sin^2 2qa + \mathrm{i}[2q(V_1 + V_2) + V_1V_2\sin 4qa]}. \quad (64)$$

† Solution of non-relativistic equation (13) with the potential $V(y) = V_1\delta(y - a) + V_2\delta(y + a)$ gives the same result.

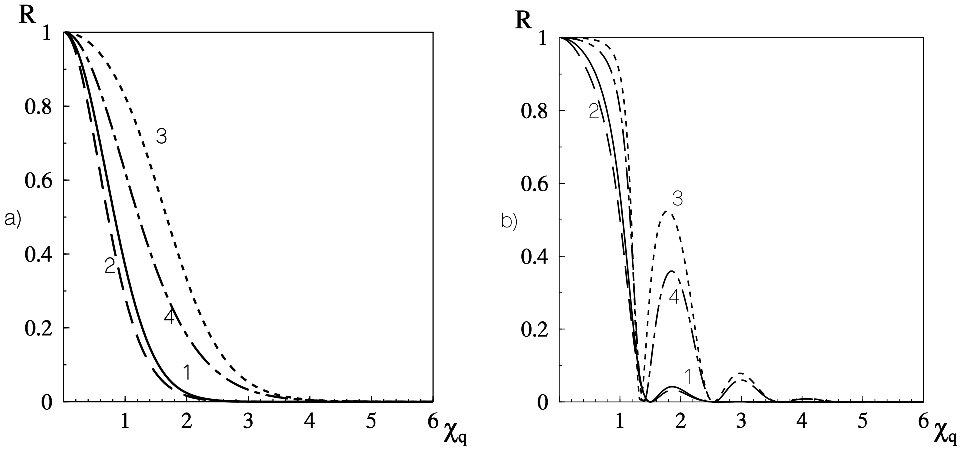

Figure 1. Reflectivity as a function of rapidity χ_q for $m = 1$, $V_1 = V_2 = -3$ (two holes) (1—$R^{(1)}$, 2—$R^{(2)}$, 3—$R^{(3)}$, 4—$R^{(4)}$): (*a*) $a = 0.1$; (*b*) $a = 1.5$.

It is easy to show that non-relativistic formulae for $A_q^{(0)}$ and $B_q^{(0)}$ ((63) and (64)) if the parameters of the potential are chosen as follows: $V_1 = V$; $V_2 = 0$, reduce to (38), derived for the simple potential (28) ($B_q^{(0)}$ coincides with it within the phase factor $\exp(\mathrm{i}2qa)$). By similar arguments the relativistic formulae for $A_q^{(j)}$ and $B_q^{(j)}$ ((54) with (49), (55) and (59)) for the same parameters of the potential reduce to (37) ($B_q^{(j)}$ coincides with it within the phase factor $\exp(\mathrm{i}2\chi_q ma)$). It means that the penetration coefficient $P_q^{(j)}$ and reflection coefficient $R_q^{(j)}$, calculated for the relativistic problem with potential (41), can be directly reduced to the corresponding ones if the potential is given by $V(\rho) = V\delta(\rho - a)$ or $V(\rho) = V\delta(\rho + a)$.

Using (63) and (64) it is not difficult to prove that the unitary relation $|1+A_q^{(0)}|^2+|B_q^{(0)}|^2 = 1$ holds. Expressions for the relativistic quantities $A_q^{(j)}$ and $B_q^{(j)}$ are more complicated than their non-relativistic analogues. But it is possible to check that the unitary relation $P_q^{(j)} + R_q^{(j)} = 1$ holds for any $j = 1$–4 (we have carried out these calculations using the algebraic programming system REDUCE).

4. Results of numerical calculations

Let us now consider the results of numerical calculations. In figures 1 and 2 the reflection coefficients $R_q^{(j)} = |B_q^{(j)}|^2$ are given as functions of the rapidity χ_q for fixed parameters a, $V_1 = V_2 = V < 0$ and $m = 1$. As we can see the reflection coefficients $R_q^{(j)}$ (for any $j = 1$–4) vanishes when $V_1 = V_2 = V$ provided that

$$2K_q^{(j)} \cos\tilde{\chi}_q + D_{Bq}^{(j)}(a)V = 0 \qquad j = 1\text{–}4. \tag{65}$$

The non-relativistic reflectivity has the same behaviour. Explicitly, formulae for the condition when the potential is completely penetrable for $j = 1, 3$ are as follows

$$\tan 2\chi_q ma = \left[\frac{2\chi_q}{\pi} - \frac{m\sinh 2\chi_q}{V}\right]\tanh \pi ma \qquad j = 1 \tag{66}$$

$$\tan 2\chi_q ma = -\frac{2m\sinh\chi_q}{V}\coth \pi ma \qquad j = 3. \tag{67}$$

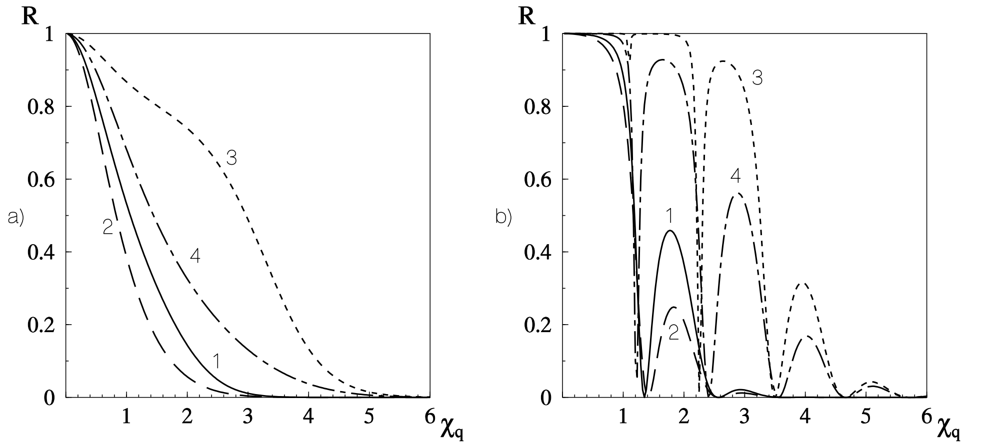

Figure 2. Reflectivity as a function of rapidity χ_q for $m = 1$, $V_1 = V_2 = -18$ (two holes): (*a*) $a = 0.1$; (*b*) $a = 1.5$.

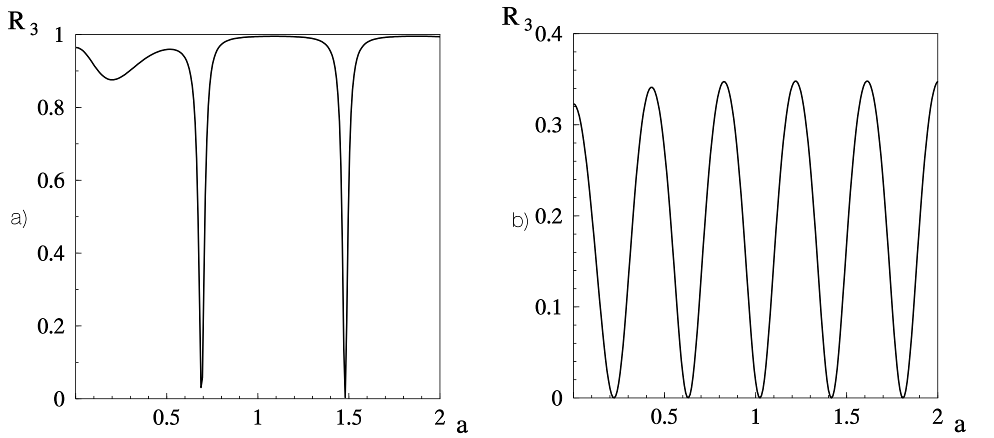

Figure 3. Reflectivity as a function of width a for $j = 3$, $V_1 = V_2 = 18$ (two barriers): (*a*) $\chi_q = 2$; (*b*) $\chi_q = 4$.

As we can see that each of these transcendental equations (65) with respect to χ_q has an infinite set of solutions $\chi_{qn}^{(j)}$. The solutions of (66) and (67) for $V < 0$ and $a > |V|^{-1} + (\pi m)^{-1}$ are equal to

$$\chi_{qn}^{(j)} = \frac{1}{2ma}\left(\frac{\pi}{2} + \pi n - \varepsilon_n\right) \qquad n = 1, 2, 3, \ldots \tag{68}$$

where $\varepsilon_n > 0$ are decreasing for large values of n. For example, for $a = 1.5$ we have $\chi_{qn}^{(j)} \cong (\pi/6)(1 + 2n)$ (see figure 1(*b*)). It should be noted here that if the parameter $|V|$ increases then the corrections ε_n increase too.

Using similar arguments we can analyse the reflectivity as a function of width a for fixed V_1, V_2, χ_q. Since the curves have a similar form for all four equations in figure 3 we plot the curves for $R_q^{(3)}$. Solving equation (67) with respect to a we again obtain an infinite set of solutions $a_n^{(3)}$. Moreover, the difference between a_{n+1} and a_n is equal to $\Delta a_n = a_{n+1} - a_n \cong \pi(2\chi_q m)^{-1}$ and if n increases the latter relation holds with a high

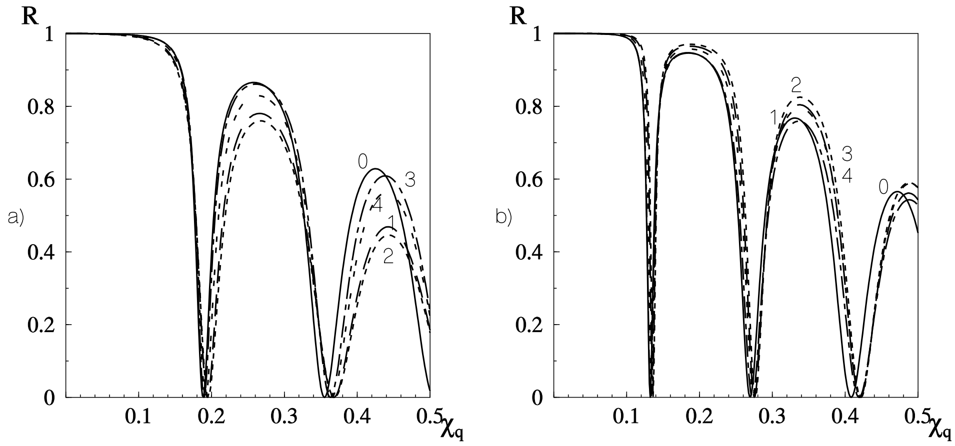

Figure 4. Reflectivity as a function of rapidity χ_q for $a = 10, m = 1, V_1 = V_2 = V_0$ (two barriers), curve 0 corresponds to the non-relativistic $R^{(0)}$: (*a*) $V_0 = -0.5$; (*b*) $V_0 = 0.5$.

accuracy (see figure 3).

The curves for $R_q^{(j)}$ have appreciably different behaviour for different j (see figures 1–3). Therefore, it seems natural to compare, in detail, the relativistic $R_q^{(j)}$ with the non-relativistic one. The point is that the curves in figures 1–3 are essentially relativistic, since $|V|m^{-1} = 3, 18$ and $0 \leqslant \chi_q \leqslant 6$. In the non-relativistic approximation, when V is fixed and $m \to \infty$, the following inequalities should be satisfied: $|V|m^{-1} \ll 1$ and $qm^{-1} \ll 1$, but this implies that $\chi_q \ll 1$. To compare the relativistic and non-relativistic results in the above-mentioned range of values let us consider the following parameters: $|V| = 0.5m$; $0 \leqslant \chi_q \leqslant 0.5$. To emphasize the descriptive behaviour of the curves let us consider a large value of the parameter a. In figure 4 the relativistic reflection coefficients $R_q^{(j)}$ (for $j = 1$–4) and non-relativistic $R_q^{(0)}$ are given as functions of the rapidity χ_q. In the region of small values of χ_q the relativistic results coincide well with the non-relativistic ones.

Let us now consider, in more detail, the cases where $V_1, V_2 > 0$ (two barriers) and $V_1 \cdot V_2 < 0$ (barrier–hole) for $|V| > m$. The corresponding curves for the reflectivity $R_q^{(j)}$ as a function of the rapidity χ_q are given in figures 5 and 6. A typical feature of these curves is the existence of points where the reflectivity is equal to unity, except for $R_q^{(3)}$ in the case of two barriers δ-potential. Numerical analysis does not, in principle, determine whether unity is approached exactly. Naturally, it is necessary to analyse the reflectivity behaviour analytically and locate the value of rapidity χ_q where the penetration coefficient vanishes, that is, the potential becomes impenetrable. This analysis has been made.

At first sight it seems impossible that the penetration $P = |1 + A|^2$ is equal to zero, since the amplitude $1 + A$ for all cases is complex-valued and vanishes only if both the imaginary and real parts are equal to zero at the same time. It seems to us that this request *a priori* cannot be satisfied by changing only the parameter $\chi_q(V_1; V_2; a$—are fixed). But, if we present the coefficient $1 + A$ as follows:

$$1 + A_q^{(j)} = \frac{\tilde{\Delta}_{(1+A)q}^{(j)}(a)}{\tilde{\Delta}_q^{(j)}(a)} \tag{69}$$

then in all the cases the equality $\mathrm{Im}\, \tilde{\Delta}_{(1+A)q}^{(j)} = 0$ holds identically. Hence, it is necessary to consider the condition for the real part of the numerator $\mathrm{Re}\, \tilde{\Delta}_{(1+A)q}^{(j)}$ only. The explicit formulae

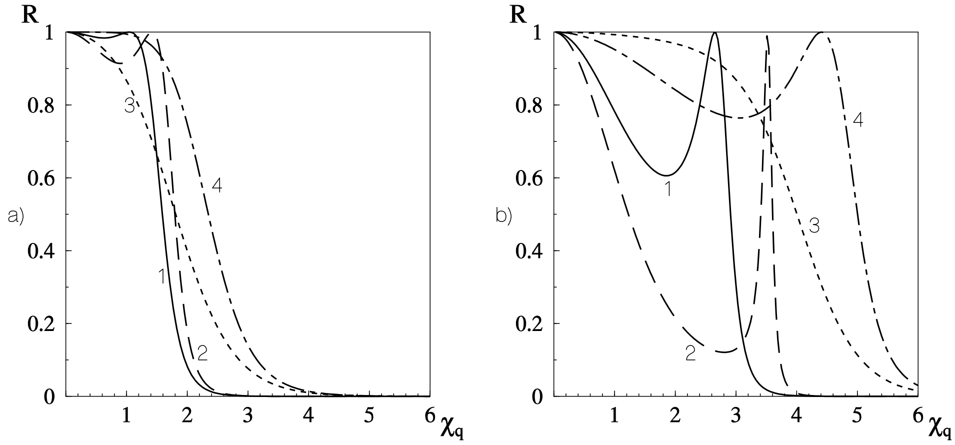

Figure 5. Reflectivity as a function of rapidity χ_q for $a = 0.05$, $m = 1$, $V_1 = V_2 = V_0$ (two barriers): (*a*) $V_0 = 3$; (*b*) $V_0 = 30$.

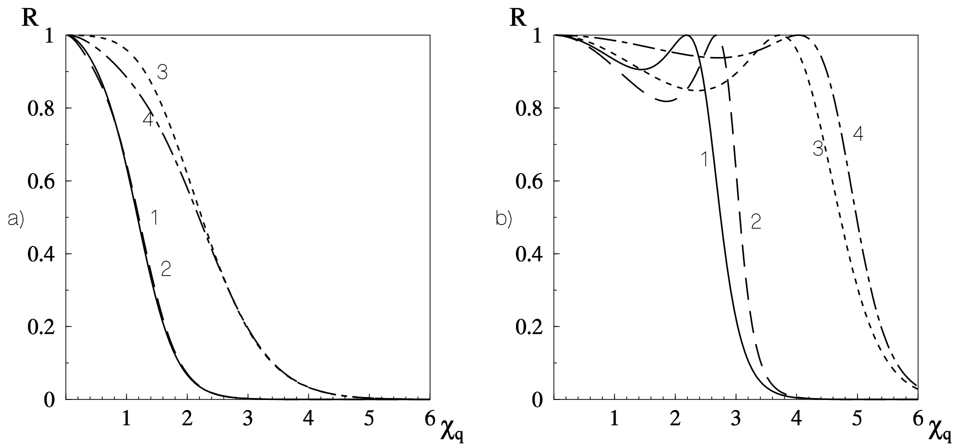

Figure 6. Reflectivity as a function of rapidity χ_q for $a = 0.2$, $m = 1$, $V_1 = -V_2 = V_0$ (barrier–hole): (*a*) $V_0 = 5$; (*b*) $V_0 = 50$.

for these conditions when the potential is impenetrable for all cases are given by

$$\prod_{s=1}^{2}\left[K_q^{(1)} - 2\frac{\chi_q}{\pi}V_s\right] - \frac{\sin^2\tilde{\chi}_q}{\sinh^2\alpha}V_1V_2 = 0 \tag{70}$$

$$\prod_{s=1}^{2}\left[K_q^{(2)} - \left(\frac{\chi_q}{\pi} + \frac{\sinh\chi_q}{2}\right)V_s\right] + \sin^2(\tilde{\chi}_q)V_1V_2$$
$$-[\coth 2\alpha\sin\tilde{\chi}_q + \sinh\chi_q(2\cosh\alpha)^{-1}]^2V_1V_2 = 0 \tag{71}$$

$$(K_q^{(3)})^2 + \frac{\sin^2\tilde{\chi}_q}{\cosh^2\alpha}V_1V_2 = 0 \tag{72}$$

$$\prod_{s=1}^{2}[K_q^{(4)} - \frac{\chi_q}{\pi}V_s] - \frac{\sin^2\tilde{\chi}_q}{\sinh^2 2\alpha}V_1V_2 = 0. \tag{73}$$

Let us now study non-trivial solutions ($\chi_q > 0$) of these equations (they do not change if we replace χ_q by $-\chi_q$). First let us consider the simplest case where $V_1 = V$; $V_2 = 0$, that is, the potential is equal to $V(\rho) = V\delta(\rho - a)$. In this case the conditions for impenetrability of the potential (70)–(73) do not contain the parameter a and are given by

$$K_q^{(j)} - \beta_q^j V = 0. \tag{74}$$

Explicitly, for $j = 1$ and $j = 4$ we have

$$\frac{\sinh 2\chi_q}{2\chi_q} = \frac{1}{\pi}\cdot\frac{V}{m} \qquad \frac{\sinh \chi_q}{\chi_q} = \frac{1}{2\pi}\cdot\frac{V}{m}. \tag{75}$$

The function $f(x) = \sinh x/x$ is monotonically increasing for $x > 0$, moreover, $f(0) = 1$. Consequently, there is no solution of the equation $f(x) = b$ provided that $b < 1$. But if $b > 1$, this equation has only one solution ($x > 0$) which we denote by $x = \varphi(b)$. Obviously, the function $\varphi(b)$ is monotonically increasing for $b > 1$ and, by the way,

$$\chi_q^{(1)}(V) = \frac{1}{2}\varphi\left(\frac{1}{\pi}\cdot\frac{V}{m}\right) \qquad \chi_q^{(4)}(V) = \varphi\left(\frac{1}{2\pi}\cdot\frac{V}{m}\right). \tag{76}$$

We carried out similar calculations for $j = 2, 3$ and for $j = 2$ obtained the same results, for $j = 3$ equation (74) has no solutions. Thus, for $j = 1, 2, 3$ and $V > V_{\min}^{(j)}$, where

$$V_{\min}^{(1)} = \pi m \qquad V_{\min}^{(2)} = 4\pi(2+\pi)^{-1}m \qquad V_{\min}^{(4)} = 2\pi m \tag{77}$$

the condition for total reflection (74) for the potential $V(\rho) = V\delta(\rho - a)$ has only one solution $\chi_q^{(j)}(V)$, moreover, $\mathrm{d}\chi_q^{(j)}(V)/\mathrm{d}V > 0$.

It should be noted here that the effect of total reflection is valid for the one-dimensional Dirac equation with scalar δ-potential (and with superposition of scalar and vector δ-potentials) as well [16]. Nevertheless, this effect only exists provided that the dimensionless coupling constant g is equal to 2: $g = 2$. It is to be emphasized that the effect of total reflection for the Dirac equation at $g = 2$ is valid for any value of rapidity (or momentum). In this paper the effect of total reflection is valid for any value of the dimensionless coupling constant V/m, if the inequality $V/m > V_{\min}^{(j)}/m$ holds, but if V/m ($V > V_{\min}$) is fixed this effect only exists for a unique value of the rapidity χ_q.

Now let us turn our attention to the general conditions for impenetrability of the potential given by the superposition of two δ-potentials. To locate the points (a, χ_q) where the potential is impenetrable let us fix the parameter V. Let us first consider the region of large values of the parameter a ($am \gg 1$), where the conditions of impenetrability can be simplified as follows

$$[K_q^{(j)} - \beta_q^j V_1][K_q^{(j)} - \beta_q^j V_2] = 0. \tag{78}$$

For $j = 3$ there is no solution of (78). For $j = 1, 2, 4$ there is solution $\chi_q^{(j)}(V_1)(\chi_q^{(j)}(V_2))$ provided that $V_1 > V_{\min}^{(j)}(V_2 > V_{\min}^{(j)})$. It means, for example, that if both parameters V_1 and V_2 satisfy the following inequality: $V_{1,2} > V_{\min}^{(j)}$, then there are two values of rapidity χ_q ($am \gg 1$) for which the effect of total reflection can be observed.

Second, let us consider the region of small values of the parameter a ($am \ll 1$) and not too large values of the rapidity χ_q. In this region equations (70)–(73) are simplified as well and for $j = 1, 2, 4$ are given by

$$K_q^{(j)} - \beta_q^j(V_1 + V_2) = 0. \tag{79}$$

The solution of (79) is equal to $\chi_q = \chi_q^{(j)}(V_1 + V_2)$ (see (76)) provided that $V_1 + V_2 > V_{\min}^{(j)}$. It means that if $am \ll 1$ the impenetrability conditions for the superposition of δ-potentials (41) is similar to the corresponding one for the potential $(V_1 + V_2)\delta(\rho)$ and this fact is physically correct.

For $j = 3$, $am \ll 1$ and not too large value of rapidity χ_q equation (72) can be written as follows

$$(K_q^{(3)})^2 + (2\chi_q ma)^2 V_1 V_2 = 0. \tag{80}$$

If $V_1 \cdot V_2 > 0$ the last equation has no solutions (and (72) in the general case as well), if $V_1 \cdot V_2 < 0$ it can be reduced to $f(\chi_q) = a\sqrt{-V_1V_2}$ and has $\chi_q = \varphi(a\sqrt{-V_1V_2})$ as a solution provided that $a\sqrt{-V_1V_2} > 1$. For $a < \sqrt{-V_1V_2}$ there is no solution of equation (80). It means that, if the parameter a tends to zero, the total reflection for the superposition of δ-potentials (41) ($j = 3$) is also absent as for the single δ-potential (28).

In the other regions of the plane (a, χ_q) each of the equations has to be studied separately. For instance, let us consider equation (70) for $\chi_q ma \ll 1$. Using the following notations ($\xi > 1$; $\zeta > 1$):

$$\xi = \frac{\sinh 2\chi_q}{2\chi_q} \qquad \zeta = \frac{\sinh \pi ma}{\pi ma} \qquad v_s = \frac{V_s}{\pi m} \tag{81}$$

equation (80) can be reduced to

$$\zeta^2 = \frac{v_1 v_2}{(\xi - v_1)(\xi - v_2)} \tag{82}$$

moreover, the region $(\xi - 1)(\zeta - 1) \ll 1$ corresponds to the region $\chi_q ma \ll 1$.

Let us consider ζ in the following limit: $\zeta = 1$. Thus, if $v_1 + v_2 > 1$ then $\xi = v_1 + v_2$ and, hence, $2\chi_q = \varphi(v_1 + v_2)$. If $v_1 + v_2 < 1$ then there is no solution. In another limit $\xi = 1$ from (82) we have

$$\zeta^2 = \zeta_0^2 = \frac{v_1 v_2}{(1 - v_1)(1 - v_2)}. \tag{83}$$

The inequality $\zeta_0^2 > 1$ is valid for one of the following cases:

(i) $v_{1,2} > 1$,
(ii) $0 < v_{1,2} < 1$; $v_1 + v_2 > 1$,
(iii) $v_1 > 1$; $v_2 < 0$; $v_1 + v_2 < 1$.

If the relation $\zeta = f(2\chi) \cong 1 + 2\chi^2/3$ holds ($\chi \ll 1$) then from (82) we obtain

$$\zeta = \zeta_0 \left[1 - \frac{\chi^2}{3}\left(\frac{1}{1 - v_1} + \frac{1}{1 - v_2}\right)\right]. \tag{84}$$

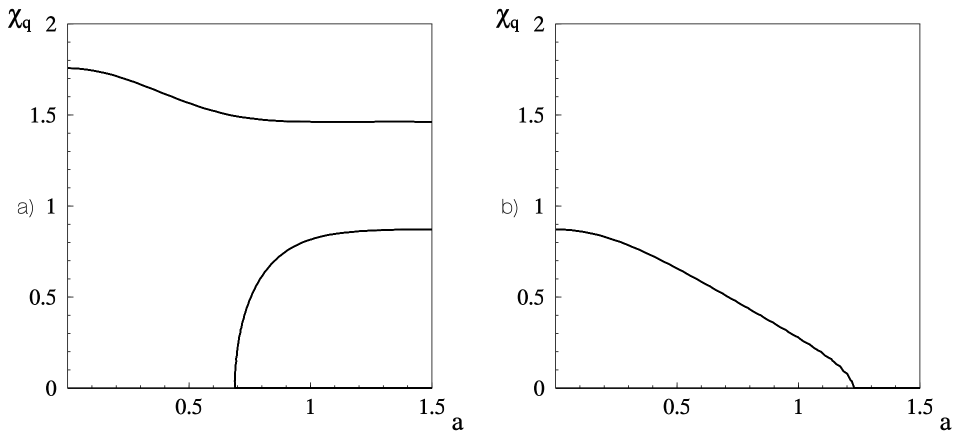

Figure 7. Solution of the condition for total reflection (70) for fixed V_1 and V_2, $m = 1$ ($j = 1$): (*a*) $V_1 = 5$, $V_2 = 10$ ($v_{1,2} < 1$); (*b*) $V_1 = 2$, $V_2 = 3$ ($v_{1,2} < 1$; $v_1 + v_2 > 1$).

In figure 7 we represent results of the numerical calculation of the impenetrability condition (70) for fixed V_1 and V_2.

To simplify the investigation of equations (70)–(73) it is necessary to fix the parameters χ_q and a and solve these equations with respect to $V_{1,2}$. For example, for the curves presented in figure 6, that is, for $V_1 = -V_2 = V_0$, it follows from (70) and (72) that

$$V_0 = \pm \frac{\pi m \sinh 2\chi_q \sinh \pi m a}{[4\chi_q^2 \sinh^2(\pi m a) - \pi^2 \sin^2(2\chi_q m a)]^{1/2}} \qquad j = 1 \tag{85}$$

$$V_0 = \pm \frac{2m \sinh \chi_q \cosh \pi m a}{\sin 2\chi_q m a} \qquad j = 3. \tag{86}$$

It means that for $j = 1$ (and for $j = 4$ as well) there is such a value of the parameter V_0 for which the total reflection is observed for any value of the 'width' a and rapidity χ_q. For $j = 3$ for all the points (a, χ_q), except for the points on the curves $\sin(2\chi_q m a) = 0$, this effect is also present. For $j = 2$ the region of values of the parameters a and χ_q, where the total reflection is absent for any value of $V_1 = -V_2$, is determined in a complicated way.

For example, let us consider (85). If $\chi_q = 2.19$, $a = 0.2$, $m = 1$ then $V = 50$. On the other hand, it follows from (86) that $V = 50$ if $\chi_q = 3.73$, $a = 0.2$, $m = 1$. This fact is in agreement with the results presented in figure 6.

Acknowledgment

TAA would like to thank the High Energy Section of ICTP for hospitality and excellent working opportunities.

References

[1] Logunov A A and Tavkhelidze A N 1963 *Nuovo Cimento* **29** 380
[2] Kadyshevsky V G 1968 *Nucl. Phys.* B **6** 125
[3] Khrustalev O A 1969 *Preprint* IHEP 69-24 (Serpukhov)
[4] Arhipov A A and Savrin V I 1982 *Preprint* IHEP 82-21 (Serpukhov)
[5] Kadyshevsky V G, Mir-Kasimov R M and Skachkov N B 1968 *Nuovo Cimento* A **55** 233
[6] Kadyshevsky V G, Mir-Kasimov R M and Skachkov N B 1972 *Sov. J. Part. Nucl.* **3** 638
[7] Kapshai V N and Skachkov N B 1983 *Sov. J. Theor. Math. Phys.* **55** 26
[8] Baz A I, Zeldovich Y B and Perelomov A M 1971 *Scattering, Reactions and Decay in Non-Relativistic Quantum Mechanic* (Moscow: Nauka) (in Russian)
[9] Demkov Y N and Ostrovsky V N 1988 *Zero-Range Potentials and their Applications in Atomic Physics* (New York: Plenum) (translation of the Russian edition: (Leningrad: Leningrad University Press))
[10] Albeverio S *et al* 1988 *Solvable Models in Quantum Mechanics* (New York: Springer)
[11] Albeverio S, Dabrowski L and Kurasov P 1998 *Lett. Math. Phys.* **45** 33
[12] Kurasov P 1996 *J. Math. Anal. Appl.* **201** 297
[13] Seba P 1986 *Czech. J. Phys.* B **36** 667
[14] Dittrich J, Exner P and Seba P 1989 *J. Math. Phys.* **30** 2875
[15] Calkin M G, Kiang D and Nogami Y 1987 *Am. J. Phys.* **55** 737
[16] Calkin M G, Kiang D and Nogami Y 1988 *Phys. Rev.* C **38** 1076
[17] Benvegnù S and Dabrowski L 1994 *Lett. Math. Phys.* **30** 159
[18] Hughes R 1997 *Rep. Math. Phys.* **39** 425
[19] Hughes R 1995 *Lett. Math. Phys.* **34** 395
[20] McKellar B H J and Stephenson G J 1987 *Phys. Rev.* C **35** 2262
[21] Subramanian R and Bhagwat K V 1972 *J. Phys. C: Solid State Phys.* **5** 798
[22] Brown G E and Jackson A D 1976 *The Nucleon–Nucleon Interaction* (Amsterdam: North-Holland) p 248